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DEPARTMENT OF NUMERICAL MATHEMATICS NW 23/76 MAY

M.R. BEST & H.J.J. TE RIELE

# ON A CONJECTURE OF ERDÖS CONCERNING SUMS OF POWERS OF INTEGERS

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# ON A CONJECTURE OF ERDÖS CONCERNING SUMS OF POWERS OF INTEGERS

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On a conjecture of Erdös concerning sums of powers of integers

by

M.R. Best & H.J.J. te Riele

ABSTRACT

It is shown here that if m and n are positive integers such that $m \geq 2$ and $(1 - \frac{1}{m-1})^n \geq \frac{1}{2}$, then

$$(*) \qquad 1^n+2^n+\ldots+(m-1)^n \geq m^n,$$

settling a conjecture of ERDÖS (Amer. Math. Monthly, 56(1949), p.343, Advanced Problem 4347).

Moreover, it is proved that the set M of integers $m \leq x$, such that there is an integer n for which $(1 - \frac{1}{m-1})^n < \frac{1}{2}$ and for which (*) holds, has cardinality $\mathcal{O}(\log x)$, for $x \to \infty$. The question whether M is finite or infinite is still open, but, by inspecting the convergents of the regular continued fraction of 2/log 2, we computed 33 elements of M.

KEY WORDS & PHRASES: *Inequalities, sums of powers of integers, continued fractions, multiple-precision arithmetic*

## 1. RESULTS

Consider pairs of integers (m,n) with $m \geq 2$, such that

$$(1.1) \qquad 1^n+2^n+\ldots+(m-1)^n \geq m^n.$$

For large m the sequence $(m-1)^n$, $(m-2)^n$, $(m-3)^n,\ldots$ can very closely be approximated by a geometrical sequence. In this way it is easily verified that (1.1) can never be satisfied if $(m-1)^n < \frac{1}{2}m^n$, (cf. VAN DE LUNE [3], particularly the addendum). On the other hand it is known that (1.1) is satisfied if $(m-3)^n > \frac{1}{2}(m-2)^n$ (cf. VAN DE LUNE, l.c.).

At first sight, numerical data strongly suggest that (1.1) is true if and only if $(m-2)^n \geq \frac{1}{2}(m-1)^n$. Indeed, VAN DE LUNE and TE RIELE [4] proved that this equivalence holds for almost all n (in the sense that the natural density equals 1). However, ERDŐS [1] had conjectured long before that (1.1) holds

i) for all pairs (m,n) with $(m-2)^n \geq \frac{1}{2}(m-1)^n$, and moreover

ii) for infinitely many pairs (m,n) with $(m-2)^n < \frac{1}{2}(m-1)^n$.

In this note, we prove the first conjecture.

THEOREM 1. *Let* m *and* n *be integers such that* $m \geq 2$ *and* $(m-2)^n \geq \frac{1}{2}(m-1)^n$. *Then* $\sum_{k=1}^{m-1} k^n \geq m^n$.

This theorem is derived as an immediate consequence of the two following theorems:

THEOREM 2. *Let* m *and* n *be integers such that* $m \geq 2$ *and* $n \geq (m - \frac{3}{2} - \frac{1}{12m})\log 2$. *Then* $(m-2)^n < \frac{1}{2}(m-1)^n$.

THEOREM 3. *Let* m *and* n *be integers such that* $m \geq 2$ *and* $n \leq (m - \frac{3}{2} - \frac{1}{256m})\log 2$. *Then* $\sum_{k=1}^{m-1} k^n \geq m^n$.

Up till now not a single pair (m,n) satisfying (1.1) and $(m-2)^n < \frac{1}{2}(m-1)^n$ was known. The theorems 2 and 3 suggest however how to construct such pairs: n/m must be a good approximation to log 2. More precisely, every pair (m,n) which satisfies the Diophantine inequality

$$(1.2) \qquad \frac{1}{128mn} \le \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1}{6mn},$$

satisfies (1.1) and $(m-2)^n < \frac{1}{2}(m-1)^n$.

By use of the convergents of the regular continued fraction of 2/log 2 we found 33 examples of such pairs, the smallest one being

m = 1 12162 60233 52385
n = 77745 19157 29368.

It seems a hopeless task to prove that (1.2) has infinitely many solutions, thus settling Erdös's second conjecture, since (1.2) is not essentially weaker than the conditions in the conjecture (cf. section 5).

In this report, some results from the theory of continued fractions are used. They are listed in the appendix.

## 2. PROOF OF THEOREM 2.

Suppose $m \ge 2$ and $n \ge (m - \frac{3}{2} - \frac{1}{12m})\log 2$.

If $m = 2$, then $n \ge \frac{11}{24}\log 2$, so $n \ge 1$, hence $(\frac{m-2}{m-1})^n = 0 < \frac{1}{2}$.

If $m = 3$, then $n \ge \frac{53}{36}\log 2$, so $n \ge 2$, hence $(\frac{m-2}{m-1})^n \le (\frac{1}{2})^2 = \frac{1}{4} < \frac{1}{2}$.

If $m = 4$, then $n \ge \frac{119}{48}\log 2$, so $n \ge 2$, hence $(\frac{m-2}{m-1})^n \le (\frac{2}{3})^2 = \frac{4}{9} < \frac{1}{2}$.

Now let $m \ge 5$. Put $\ell = m - 1$. Then $\ell \ge 4$ and

$$n \ge (\ell - \tfrac{1}{2} - \frac{1}{12(\ell+1)})\log 2 > (\ell - \tfrac{1}{2} - \frac{1}{12\ell})\log 2.$$

Hence

$$(\frac{m-2}{m-1})^n = (\frac{\ell-1}{\ell})^n = \exp(n \log(1 - \frac{1}{\ell})) <$$

$$< \exp((\ell - \tfrac{1}{2} - \frac{1}{12\ell})(- \frac{1}{\ell} - \frac{1}{2\ell^2} - \frac{1}{3\ell^3} - \frac{1}{4\ell^4})\log 2) =$$

$$= \exp(-(1 + \frac{1}{24\ell^3} - \frac{11}{72\ell^4} - \frac{1}{48\ell^5})\log 2) =$$

$$= \exp(-(1 + \frac{1}{144\ell^5}(6\ell^2 - 22\ell - 3))\log 2) < \exp(-\log 2) = \tfrac{1}{2}.$$

This completes the proof. □

## 3. PROOF OF THEOREM 3.

In this section we shall put $\lambda = \log 2$ and $S = \sum_{k=1}^{m} (\frac{k}{m})^n$. Now Theorem 3 states that

$$S \geq 2$$

provided that

$$n \leq \lambda(m - \frac{3}{2} - \frac{1}{256m}).$$

In order to prove this, we need some identities concerning the functions $f_j$ defined by

$$f_j(u) = \sum_{k=0}^{\infty} k^j e^{-ku} \quad (u > 0,\ j = 0,1,2,\ldots).$$

LEMMA. *We have* $f_j = -f'_{j-1}$, *for* $j = 1,2,\ldots$. *If* $x = e^{-u}$ *then*

$$f_0(u) = \frac{1}{1-x}, \qquad f_1(u) = \frac{x}{(1-x)^2}, \qquad f_2(u) = \frac{x+x^2}{(1-x)^3},$$

$$f_3(u) = \frac{x+4x^2+x^3}{(1-x)^4}, \qquad f_4(u) = \frac{x+11x^2+11x^3+x^4}{(1-x)^5},$$

$$f_5(u) = \frac{x+26x^2+66x^3+26x^4+x^5}{(1-x)^6}; \quad f_6(u) = \frac{x+57x^2+302x^3+302x^4+57x^5+x^6}{(1-x)^7},$$

$$f_0(\lambda) = f_1(\lambda) = 2,\ f_2(\lambda) = 6,\ f_3(\lambda) = 26 \text{ and } f_4(\lambda) = 150.$$

PROOF. Straightforward. □

REMARK. The coefficients in the polynomials occurring above are the Eulerian numbers (cf. RIORDAN [6]).

To prove the theorem, it suffices to consider $n = \lfloor \lambda(m - \frac{3}{2} - \frac{1}{256m}) \rfloor$. 1) We have:

$$S = \sum_{k=0}^{m-1} (1 - \frac{k}{m})^n \geq \sum_{k=0}^{\ell} \exp(n \log(1 - \frac{k}{m})),$$

1) By $\lfloor x \rfloor$ we mean the greatest integer less than or equal to x.

where $\ell < m$ may be chosen arbitrarily. Since for $0 \le x < 1$:

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \ge - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4(1-x)} ,$$

we have

$$S \ge \sum_{k=0}^{\ell} \exp(n(-\frac{k}{m} - \frac{k^2}{2m^2} - \frac{k^3}{3m^3} - \frac{k^4}{4m^3(m-\ell)})) =$$

$$= \sum_{k=0}^{\ell} \exp(- kt - k^2\varepsilon - k^3\eta - k^4\theta)$$

where $t = \frac{n}{m}$, $\varepsilon = \frac{n}{2m^2}$, $\eta = \frac{n}{3m^3}$, and $\theta = \frac{n}{4m^3(m-\ell)}$ .

Defining $\rho = \sum_{k>\ell} e^{-kt}$, we have

$$S \ge \sum_{k=0}^{\infty} \exp(- kt - k^2\varepsilon - k^3\eta - k^4\theta) - \rho \ge$$

$$\ge \sum_{k=0}^{\infty} e^{-kt}(e^{-k^2\varepsilon} - (k^3\eta + k^4\theta)e^{-k^2\varepsilon}) - \rho \ge$$

$$\ge \sum_{k=0}^{\infty} e^{-kt}(1 - k^2\varepsilon + \tfrac{1}{2}k^4\varepsilon^2 - \tfrac{1}{6}k^6\varepsilon^3 - k^3\eta - k^4\theta) - \rho =$$

$$= f_0(t) - \varepsilon f_2(t) - \eta f_3(t) + \tfrac{1}{2}\varepsilon^2 f_4(t) - \theta f_4(t) - \tfrac{1}{6}\varepsilon^3 f_6(t) - \rho.$$

Now define $\nu$ by $n = \lambda(m-\nu)$, so $\nu = m - n/\lambda$. Since

$$\lambda(m - \frac{3}{2} - \frac{1}{256m}) > n > \lambda(m - \frac{3}{2} - \frac{1}{\lambda} - \frac{1}{256m}),$$

it follows that

$$\frac{3}{2} + \frac{1}{256m} \le \nu < \frac{3}{2} + \frac{1}{\lambda} + \frac{1}{256m} < 2.95 + \frac{1}{256m},$$

hence

$$\frac{3}{2} \le \nu \le 3.$$

Now assume $m > 700$, and choose $\ell = 5 \log m$. Then by the lemma we have

$$\lambda > t = \frac{n}{m} = \lambda(1 - \frac{\nu}{m}) > \lambda(1 - \frac{3}{700}) > 0.69,$$

$$\varepsilon = \frac{\lambda}{2m} - \frac{\lambda\nu}{2m^2}, \qquad \eta = \frac{\lambda}{3m^2} - \frac{\lambda\nu}{3m^3} < \frac{\lambda}{3m^2},$$

$$\theta = \frac{\lambda(1 - \frac{\nu}{m})}{4m^3(1 - \frac{\ell}{m})} < \frac{\lambda}{4m^3(1 - \frac{5 \log 700}{700})} < \frac{0.27\lambda}{m^3},$$

$$\tfrac{1}{2}\varepsilon^2 > \frac{\lambda^2}{8m^2} - \frac{\lambda^2\nu}{4m^3} > \frac{\lambda^2}{8m^2} - \frac{0.52\lambda}{m^3}, \qquad \tfrac{1}{6}\varepsilon^3 < \frac{\lambda^3}{48m^3} < \frac{0.01001\lambda}{m^3},$$

$$\rho < \frac{e^{-\ell t}}{1-e^{-t}} < \frac{m^{-3.45}}{0.49} < \frac{\lambda}{m^3},$$

$$f_4(t) < f_4(0.69) < 154, \qquad f_6(t) < f_6(0.69) < 9670,$$

$$f_0(t) = f_0(\lambda) + \frac{\lambda\nu}{m} f_1(\lambda) + \frac{\lambda^2\nu^2}{2m^2} f_2(\xi) > 2 + \frac{2\lambda\nu}{m} + \frac{3\lambda^2\nu^2}{m^2} \qquad (t < \xi < \lambda),$$

$$f_2(t) = f_2(\lambda) + \frac{\lambda\nu}{m} f_3(\lambda) + \frac{\lambda^2\nu^2}{2m^2} f_4(\xi) < 6 + \frac{26\lambda\nu}{m} + \frac{154\lambda^2\nu^2}{2m^2} <$$

$$< 6 + \frac{26\lambda\nu}{m} + \frac{333}{m^2} \qquad (t < \xi < \lambda),$$

$$f_3(t) = f_3(\lambda) + \frac{\lambda\nu}{m} f_4(\xi) < 26 + \frac{154\lambda\nu}{m} < 26 + \frac{321}{m} \qquad (t < \xi < \lambda),$$

$$f_4(t) > f_4(\lambda) = 150.$$

Substituting all these estimates in the estimate for $S$, we obtain:

$$S > 2 + \frac{2\lambda\nu}{m} + \frac{3\lambda^2\nu^2}{m^2} - (\frac{\lambda}{2m} - \frac{\lambda\nu}{2m^2})(6 + \frac{26\lambda\nu}{m} + \frac{333}{m^2}) - \frac{\lambda}{3m^2}(26 + \frac{321}{m}) +$$

$$+ (\frac{\lambda^2}{8m^2} - \frac{0.52\lambda}{m^3})150 - \frac{0.27\lambda}{m^3}154 - \frac{0.01001\lambda}{m^3}9670 - \frac{\lambda}{m^3} >$$

$$> 2 + \frac{2\lambda\nu}{m} + \frac{3\lambda^2\nu^2}{m^2} - \frac{3\lambda}{m} - \frac{13\lambda^2\nu}{m^2} - \frac{167\lambda}{m^3} + \frac{3\lambda\nu}{m^2} - \frac{26\lambda}{3m^2} - \frac{107\lambda}{m^3} +$$

$$+ \frac{75\lambda^2}{4m^2} - \frac{78\lambda}{m^3} - \frac{42\lambda}{m^3} - \frac{97\lambda}{m^3} - \frac{\lambda}{m^3} =$$

$$= 2 + \frac{\lambda}{m}(2\nu-3) + \frac{\lambda}{m^2}(3\lambda\nu^2 - (13\lambda-3)\nu + \frac{75\lambda}{4} - \frac{26}{3}) - \frac{492\lambda}{m^3}.$$

Since $3\lambda\nu^2 - (13\lambda-3)\nu$ is a monotonically increasing function of $\nu$ for $\nu \geq \frac{13\lambda-3}{6\lambda} = 1.445...$, we derive from $\nu > \frac{3}{2}$:

$$S > 2 + \frac{\lambda}{m}(2\nu-3) + \frac{\lambda}{m^2}(\frac{27\lambda}{4} - \frac{39\lambda}{2} + \frac{9}{2} + \frac{75\lambda}{4} - \frac{26}{3}) - \frac{492\lambda}{m^3} =$$

$$= 2 + \frac{\lambda}{m}(2\nu-3) + \frac{\lambda}{m^2}(6\lambda - \frac{25}{6} - \frac{492}{m}).$$

Now define $\mu$ by $\nu = \frac{3}{2} + \frac{\mu}{m}$, so $\mu = (\nu - \frac{3}{2})m$. Then $\mu > \frac{1}{256}$ and

$$S > 2 + \frac{\lambda}{m^2}(2\mu + 6\lambda - \frac{25}{6} - \frac{492}{m}) > 2 + \frac{\lambda}{m^2}(2\mu - 0.007784 - \frac{492}{m}) >$$

$$> 2 + \frac{\lambda}{m^2}(0.000028 - \frac{492}{m}) > 2,$$

if $m > 2*10^7$.

Thus we have proved Theorem 3 in case $m > 2*10^7$. For $m \leq 700$ the theorem has been checked by direct (ccmputer-) verification. Hence we may assume $700 < m \leq 2*10^7$.

First suppose $\mu \geq \frac{1}{4\lambda}$. Then

$$S > 2 + \frac{\lambda}{m^2}(2\mu - 0.007784 - \frac{492}{m}) > 2 + \frac{\lambda}{m^2}(0.72 - 0.01 - \frac{492}{700}) > 2.$$

Hence we may assume moreover that $\mu < \frac{1}{4\lambda}$. But then by the definition of $\nu$ and $\mu$:

$$\mu = (\nu - \frac{3}{2})m = (m - \frac{n}{\lambda} - \frac{3}{2})m = (\frac{2m-3}{n} - \frac{2}{\lambda})\frac{mn}{2},$$

hence

$$0 < \frac{2m-3}{n} - \frac{2}{\lambda} = \frac{2\mu}{mn} < \frac{1}{2mn\lambda} < \frac{1}{2n^2}.$$

This implies that $(2m-3)/n$ is a convergent of the regular continued fraction of $2/\log 2$ (see the appendix, (A7)). There are only three convergents $p_k/q_k$ which come into consideration, i.e., for which $p_k/q_k - 2/\log 2 > 0$, $p_k$ is odd and $700 < m \le 2*10^7$:

i) $2m - 3 = p_9 = 2291$, $n = q_9 = 794$, $m = 1147$, $b_{10} = 4$;
ii) $2m - 3 = p_{13} = 1206321$, $n = q_{13} = 418079$, $m = 603162$, $b_{14} = 6$;
iii) $2m - 3 = p_{15} = 31668469$, $n = q_{15} = 10975455$, $m = 15834236$, $b_{16} = 1$.

Here $b_i$ is the i-th partial denominator of the regular continued fraction of $2/\log 2$. (The first 601 b's are given in Table 1 of Section 4).

Case i) has been verified directly.

Since, by (A5) of the appendix,

$$\mu > \frac{n^2}{2\lambda}\left(\frac{2m-3}{n} - \frac{2}{\lambda}\right) = \frac{n^2}{2\lambda}\left(\frac{p_k}{q_k} - \frac{2}{\lambda}\right) > \frac{n^2}{2\lambda}\,\frac{1}{(b_{k+1}+2)q_k^2} = \frac{1}{2\lambda(b_{k+1}+2)},$$

we have in the cases ii) and iii):

$$\mu > \frac{1}{16\lambda} > 0.09,$$

so that

$$S > 2 + \frac{\lambda}{m^2}\left(0.18 - 0.01 - \frac{492}{m}\right) > 2.$$

This completes the proof of Theorem 3. □

## 4. COMPUTER CALCULATIONS OF THE PAIRS (m,n) SATISFYING (1.1) AND $(m-2)^n < \frac{1}{2}(m-1)^n$.

In this section we shall describe how we have computed 33 pairs of integers (m,n) satisfying (1.1) and $(m-2)^n < \frac{1}{2}(m-1)^n$. Up till now not a single such pair was known, although Erdős conjectured that there are infinitely many of them ([1]).

It follows from Theorems 2 and 3 that in order to find such pairs, it is sufficient to find pairs (m,n) satisfying

$$\left(m - \frac{3}{2} - \frac{1}{12m}\right)\log 2 < n < \left(m - \frac{3}{2} - \frac{1}{256m}\right)\log 2,$$

or, after some reordering,

$$(4.1)\qquad \frac{1}{128mn} < \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1}{6mn}.$$

Define, as in Section 3, $\mu$ by $n = (m - \frac{3}{2} - \frac{\mu}{m})\log 2$. Then we have $\mu = \mu(m,n) = m(m - \frac{n}{\log 2} - \frac{3}{2})$ and (4.1) is equivalent with

$$(4.2)\qquad \frac{1}{256} < \mu < \frac{1}{12}.$$

Since $m > n$, (4.1) implies, by (A7), that $(2m-3)/n$ is a convergent of the regular continued fraction of $2/\log 2$. Now let $p_k/q_k$ be the k-th convergent of $2/\log 2$. Suppose k is odd. Then by (A4) $p_k/q_k - 2/\log 2 > 0$. If, moreover, $p_k$ is odd, then m and n defined by $m = (p_k+3)/2$ and $n = q_k$ satisfy (4.1) if and only if $\mu = \mu(m,n)$ satisfies (4.2).

In order to compute $p_k$ and $q_k$ $(k = 0,1,2,\ldots)$, and, if appropriate, n,m and $\mu$, we have used the ALGOL 60 procedures for multiple-precision integer arithmetic from the NUMAL-library [2]. First, the 601 partial denominators $b_0, b_1, \ldots, b_{600}$ of the regular continued fraction of $2/\log 2$ were computed from the first 700 decimals of log 2, as given by SWEENEY [7]. The b's were checked by use of (A8) of the appendix; they are given in Table 1 below.

TABLE 1.

The first 601 partial denominators of the regular continued fraction of $2/\log 2$. So $2/\log 2 = [2,1,7,1,2,1,1,\ldots]$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | |
| 1 | 7 | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 4 |
| 7 | 5 | 3 | 6 | 4 | 1 | 1 | 4 | 1 | 1 |
| 27 | 3 | 1 | 1 | 1 | 1 | 4 | 1 | 3 | 4 |
| 2 | 3 | 2 | 1 | 2 | 29 | 1 | 4 | 1 | 9 |
| 1 | 36 | 1 | 1 | 10 | 1 | 2 | 1 | 2 | 1 |
| 3 | 6 | 1 | 1 | 27 | 1 | 1 | 9 | 2 | 2 |
| 1 | 1 | 4 | 5 | 8 | 1 | 1 | 1 | 2 | 1 |
| 65 | 4 | 1 | 1 | 2 | 2 | 11 | 10 | 1 | 1 |
| 18 | 4 | 3 | 1 | 3 | 3 | 4 | 3 | 2 | 10 |
| 2 | 65 | 1 | 9 | 5 | 105 | 21 | 1 | 3 | 3 |
| 1 | 2 | 7 | 14 | 4 | 19 | 1 | 4 | 1 | 56 |
| 4 | 6 | 7 | 1 | 6 | 5 | 13 | 1 | 3 | 5 |
| 35 | 1 | 5 | 7 | 3 | 1 | 1 | 2 | 2 | 5 |
| 6 | 3 | 4 | 1 | 5 | 6 | 2 | 1 | 3 | 1 |
| 2 | 2 | 2 | 2 | 242 | 1 | 6 | 6 | 4 | 1 |
| 6 | 1 | 1 | 2 | 2 | 15 | 1 | 7 | 1 | 2 |
| 1 | 4 | 2 | 22 | 3 | 5 | 1 | 1 | 1 | 2 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 13 | 1 | 3 | 3 | 1 | 2 | 4 | 1 |
| 1 | 1 | 3 | 3 | 2 | 1 | 2 | 1 | 1 | 6 |
| 8 | 4 | 3 | 3 | 5 | 10 | 3 | 7 | 2 | 4 |
| 24 | 6 | 1 | 2 | 1 | 3 | 9 | 7 | 4 | 5 |
| 1 | 9 | 1 | 3 | 1 | 1 | 4 | 1 | 2 | 2 |
| 1 | 1 | 5 | 2 | 1 | 2 | 5 | 2 | 18 | 1 |
| 3 | 1 | 9 | 4 | 1 | 2 | 8 | 2 | 6 | 1 |
| 6 | 6 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 |
| 3 | 1 | 1 | 2 | 1 | 10 | 1 | 1 | 14 | 5 |
| 1 | 19 | 1 | 14 | 1 | 1 | 2 | 19 | 1 | 13 |
| 3 | 5 | 3 | 1 | 108 | 1 | 1 | 1 | 1 | 67 |
| 1 | 1 | 22 | 1 | 1 | 3 | 3 | 1 | 2 | 2 |
| 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 18 |
| 155 | 1 | 8 | 1 | 2 | 3 | 1 | 1 | 2 | 2 |
| 1 | 4 | 1 | 8 | 2 | 7 | 8 | 13 | 2 | 4 |
| 3 | 8 | 1 | 1 | 1 | 2 | 1 | 1 | 99 | 1 |
| 10 | 4 | 1 | 2 | 2 | 3 | 2 | 1 | 9 | 1 |
| 7 | 2 | 2 | 1 | 31 | 1 | 4 | 2 | 1 | 1 |
| 2 | 2 | 4 | 1 | 4 | 1 | 9 | 1 | 11 | 1 |
| 9 | 1 | 2 | 1 | 1 | 1 | 4 | 1 | 1 | 21 |
| 12 | 1 | 4 | 1 | 7 | 1 | 5 | 2 | 1 | 1 |
| 4 | 3 | 8 | 3 | 29 | 2 | 3 | 1 | 1 | 1 |
| 1 | 3 | 4 | 1 | 8 | 4 | 21 | 7 | 1 | 8 |
| 1 | 2 | 1 | 2 | 1 | 40 | 1 | 2 | 1 | 7 |
| 2 | 4 | 2 | 1 | 6 | 98 | 2 | 30 | 9 | 1 |
| 1 | 5 | 2 | 1 | 1 | 1 | 9 | 1 | 1 | 1 |
| 7 | 8 | 1 | 5 | 1 | 1 | 6 | 3 | 1 | 30 |
| 1 | 1 | 1 | 40 | 2 | 1 | 2 | 14 | 1 | 4 |
| 3 | 2 | 4 | 13 | 1 | 2 | 2 | 1 | 7 | 1 |
| 2 | 2 | 1 | 1 | 1 | 15 | 1 | 43 | 14 | 2 |
| 3 | 4 | 3 | 2 | 4 | 1 | 2 | 1 | 1 | 2 |
| 3 | 78 | 1 | 2 | 1 | 43 | 1 | 1688 | 2 | 2 |
| 7 | 2 | 37 | 3 | 1 | 3 | 3 | 12 | 5 | 2 |
| 14 | 2 | 3 | 1 | 1 | 2 | 47 | 14 | 14 | 1 |
| 1 | 1 | 1 | 7 | 2 | 21 | 1 | 1 | 1 | 1 |
| 3 | 1 | 2 | 1 | 15 | 2 | 6 | 5 | 6 | 3 |
| 24 | 1 | 1 | 1 | 1 | 4 | 7 | 5 | 1 | 2 |
| 9 | 1 | 526 | 1 | 1 | 5 | 1 | 1 | 6 | 6 |
| 1 | 2 | 5 | 3 | 1 | 4 | 3 | 1 | 12 | 1 |
| 2 | 4 | 2 | 2 | 3 | 3 | 1 | 1 | 1 | 1 |
| 4 | 8 | 1 | 1 | 12 | 10 | 1 | 2 | 1 | 1 |
| 2 | 6 | 6 | 14 | 3 | 5 | 1 | 1 | 15 | 1 |
| 89 | 5 | 2 | 3 | 13 | 3 | 2 | 2 | 110 | 1 |

Next, for $k = 0,1,\ldots,599$, the exact integer values of the nominator $p_k$ and the denominator $q_k$ of the k-th convergent $p_k/q_k$ of $2/\log 2$ were computed by use of the relations (A1) of the appendix. In case of odd k and odd $p_k$ the value of $\mu = \mu(m,n) = \mu((p_k+3)/2, q_k)$ was computed. In Table 2 below the odd values of $k \le 599$ are listed for which $p_k$ is odd and for which $\mu$ satisfies (4.2). For all these values of k the numbers $m = (p_k+3)/2$ and $n = q_k$ satisfy (1.1) and $(m-2)^n < \frac{1}{2}(m-1)^n$. For reasons which will become clear soon, also the corresponding values of $b_{k+1}$ are included in this table.

TABLE 2.

Odd values of k for which $p_k$ is odd and $\mu$ satisfies (4.2). ($\mu$ is rounded to five decimals).

| k | $\mu$ | $b_{k+1}$ | k | $\mu$ | $b_{k+1}$ | k | $\mu$ | $b_{k+1}$ |
|---|---|---|---|---|---|---|---|---|
| 35 | 0.02388 | 29 | 211 | 0.06792 | 9 | 439 | 0.02296 | 30 |
| 39 | 0.06676 | 9 | 255 | 0.06420 | 10 | 443 | 0.01758 | 40 |
| 41 | 0.01927 | 36 | 261 | 0.03473 | 19 | 447 | 0.04750 | 14 |
| 57 | 0.07266 | 9 | 267 | 0.03549 | 19 | 453 | 0.05180 | 13 |
| 77 | 0.06805 | 10 | 299 | 0.03870 | 18 | 467 | 0.01639 | 43 |
| 89 | 0.06600 | 10 | 313 | 0.07763 | 8 | 481 | 0.00913 | 78 |
| 91 | 0.01087 | 65 | 317 | 0.05315 | 13 | 485 | 0.01612 | 43 |
| 95 | 0.00685 | 105 | 321 | 0.08067 | 8 | 497 | 0.05776 | 12 |
| 157 | 0.08313 | 7 | 369 | 0.03335 | 21 | 507 | 0.05120 | 14 |
| 163 | 0.03168 | 22 | 399 | 0.07507 | 8 | 575 | 0.06678 | 10 |
| 195 | 0.06865 | 10 | 431 | 0.08035 | 8 | 583 | 0.04983 | 14 |

In Table 3 below we have listed the decimal representations of the numbers n and m corresponding to the cases k = 35,39,41 and 57 of Table 2. The integers n and m corresponding to k = 583 (our largest computed case) are 302-digit numbers. Their first and last five digits are given by
n = 19305...16252, m = 27852...10488.

TABLE 3.

Pairs of integers (m,n) satisfying (1.1) and $(m-2)^n < \frac{1}{2}(m-1)^n$, corresponding to the cases k = 35,39,41 and 57 of Table 2.

| | | |
|---|---|---|
| k = 35 | n = | 77745 19157 29368 |
| | m = | 1 12162 60233 52385 |
| k = 39 | n = | 140 89409 20558 57794 |
| | m = | 203 26720 78995 39136 |
| k = 41 | n = | 1526 22308 86191 71207 |
| | m = | 2201 87448 12411 15228 |
| k = 57 | n = | 5 45458 11706 25883 69110 39145 |
| | m = | 7 86929 72049 88279 15993 33820 |

The next theorem provides a partial check of Table 2.

THEOREM 4. *Let* $k(\geq 5)$ *and* $p_k$ *be odd. If* $b_{k+1}$ *satisfies*

(4.3) $$9 \leq b_{k+1} \leq 182$$

*then* m *and* n *satisfy* (4.1).

PROOF. In Table 1 we see that $b_k < 9$ for $0 \leq k \leq 20$. Hence we certainly may assume that $m > 700$, so that $0.69 < n/m < \log 2$ (cf. the proof of Theorem 3). Now by the right hand inequality of (A5) and since $b_{k+1} \geq 9$ we have

$$\frac{p_k}{q_k} - \frac{2}{\log 2} < \frac{1}{b_{k+1}q_k^2} \leq \frac{1}{9n^2} < \frac{1}{9n*0.69m} < \frac{1}{6nm}.$$

On the other hand, we have

$$\frac{p_k}{q_k} - \frac{2}{\log 2} > \frac{1}{(b_{k+1}+2)q_k^2} \geq \frac{1}{184n^2} > \frac{1}{184nm\log 2} > \frac{1}{128nm}.$$

Hence $m = (p_k+3)/2$ and $n = q_k$ satisfy the inequalities (4.1). □

5. DISCUSSION.

The main results of this note are Theorem 1 and the list of examples of pairs of integers (m,n) with $m \geq 2$, $(m-2)^n < \frac{1}{2}(m-1)^n$ and $\sum_{k=1}^{m-1} k^n \geq m^n$. Unproved however remained the following assertions.

i) The density of the integers m occurring in these examples is zero.
ii) There actually are infinitely many examples.
iii) The examples listed are smallest possible.

The first assertion was proved by VAN DE LUNE and TE RIELE [4] by a different approach. An even stronger result follows from the next two theorems:

THEOREM 5. *Let for each integer* $m \geq 2$ *the real number* n *be defined by* $(m-2)^n = \frac{1}{2}(m-1)^n$. *Then* $n = (m - \frac{3}{2} - (\frac{1}{12} + o(1))m^{-1})\log 2$ *for* $m \to \infty$.

THEOREM 6. *Let for each integer* $m \geq 2$ *the real number* n *be defined by* $\sum_{k=1}^{m-1} k^n = m^n$. *Then* $n = (m - \frac{3}{2} - (\frac{25}{12} - 3\log 2 + o(1))m^{-1})\log 2$ *for* $m \to \infty$.

The proofs of the Theorems 5 and 6 run very similar to those of Theorems 2 and 3 (except for the technical details), and will be omitted therefore. From Theorems 5 and 6 we derive:

THEOREM 7. *Let*

$$M = \{m \mid m \in \mathbb{Z} \wedge m \geq 2 \wedge \exists n \in \mathbb{Z}\,((m-2)^n < \tfrac{1}{2}(m-1)^n \wedge \sum_{k=1}^{m-1} k^n \geq m^n)\}.$$

*Then the number of integers* $m \in M$ *with* $m \leq x$ *is* $O(\log x)$, *for* $x \to \infty$.

PROOF. For all $m \in M$:

$$(m - \frac{3}{2} - (\frac{1}{12} + o(1))m^{-1})\log 2 < n < (m - \frac{3}{2} - (\frac{25}{12} - 3\log 2 + o(1))m^{-1})\log 2.$$

so

$$\frac{1}{nm}(\frac{25}{6} - 6\log 2 + o(1)) < \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1}{nm}(\frac{1}{6} + o(1)).$$

Hence, for all sufficiently large $m \in M$, we have

$$0 < \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1}{2n^2},$$

so that $(2m-3)/n$ is a convergent of the regular continued fraction of $2/\log 2$ (by (A7)). Now with increasing k the nominator and the denominator of the k-th convergent of a regular continued fraction do not increase slower than the Fibonacci sequence (cf. (A1) with $b_k = 1$). From this, the theorem follows easily. □

From Theorems 5 and 6 it also follows that the second assertion - the only unsettled part of Erdös's conjecture - is very hard to prove. If ii) were true, it would follow that for each $\varepsilon > 0$ there are infinitely many m and n such that

$$(m - \frac{3}{2} - (\frac{1}{12} + \varepsilon)m^{-1})\log 2 < n < (m - \frac{3}{2} - \frac{1}{257m})\log 2,$$

or

$$\frac{2}{257mn} < \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1/6+2\varepsilon}{mn},$$

which means that there are infinitely many partial denominators $b_k$ of $2/\log 2$ satisfying $7 \leq b_k \leq 185$ (compare 4.1 and 4.3). And this is, although quite probable, very hard to prove.

As to the assertion (iii), we only mention that we are convinced that our examples are the smallest possible ones, and that it may be proved by establishing effective forms of Theorems 5 and 6 (like Theorems 2 and 3, but all estimates to the other side), leaving out only a reasonable number of cases to be checked directly.

## APPENDIX. SOME RESULTS FROM THE THEORY OF CONTINUED FRACTIONS.

Some results from the theory of continued fractions, used in this report, are listed here. They can all be found, explicitly or implicitly, in the second chapter of [5].

Let $\alpha$ be some irrational number. The regular continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \dots}}$$

of $\alpha$ is denoted by $[b_0, b_1, b_2, \dots]$. $\alpha = [b_0, b_1, \dots, b_{k-1}, \alpha_k]$ implies that

$$\alpha_k = b_k + \cfrac{1}{b_{k+1} + \cfrac{1}{b_{k+2} + \dots}} \qquad (k = 0,1,2,\dots).$$

The numbers $b_i$ are called the partial denominators of the regular continued fraction of $\alpha$. By $p_k/q_k$ $(k = 0,1,2,\dots)$ we shall denote the k-th convergent $[b_0, b_1, b_2, \dots, b_k]$ of (the regular continued fraction of) $\alpha$. We have

$$\text{(A1)} \qquad \begin{cases} p_k = b_k p_{k-1} + p_{k-2} \\ q_k = b_k q_{k-1} + q_{k-2} \end{cases}, \quad k = 0,1,2,\dots, \text{ where } \quad \begin{matrix} p_{-1} = q_{-2} = 1, \\ p_{-2} = q_{-1} = 0. \end{matrix}$$

Furthermore, for $k = 1,2,\dots,$

$$\text{(A2)} \qquad p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}, \text{ and}$$

$$\text{(A3)} \qquad q_k/q_{k-1} = [b_k, b_{k-1}, \dots, b_2, b_1].$$

Moreover,

$$\text{(A4)} \qquad \frac{p_k}{q_k} \begin{matrix} < \\ > \end{matrix} \alpha \text{ if } k \text{ is } \begin{matrix} \text{even} \\ \text{odd} \end{matrix}.$$

From $\alpha = [b_0,b_1,\ldots,b_k,\alpha_{k+1}]$ it follows that $\alpha = \dfrac{\alpha_{k+1}p_k + p_{k-1}}{\alpha_{k+1}q_k + q_{k-1}}$, so that

$$\frac{p_k}{q_k} - \alpha = \frac{p_k q_{k-1} - q_k p_{k-1}}{q_k^2(\alpha_{k+1} + q_{k-1}/q_k)},$$

hence

$$\left|\frac{p_k}{q_k} - \alpha\right| = \frac{1}{q_k^2(b_{k+1} + [0,b_{k+2},b_{k+3},\ldots] + [0,b_k,b_{k-1},\ldots,b_1])}.$$

This equality implies the well-known inequality

(A5) $$\frac{1}{q_k^2(b_{k+1}+2)} < \left|\frac{p_k}{q_k} - \alpha\right| < \frac{1}{q_k^2 b_{k+1}} \qquad (k = 0,1,2,\ldots),$$

but also the sharper inequality

(A6) $$\frac{1}{q_k^2(b_{k+1} + \frac{1}{b_{k+2}} + \frac{1}{b_k})} < \left|\frac{p_k}{q_k} - \alpha\right| < \frac{1}{q_k^2(b_{k+1} + \frac{1}{b_{k+2}+1} + \frac{1}{b_k+1})}$$
$$(k = 0,1,2,\ldots)$$

and so on.

(A7) If the rational number p/q satisfies

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{2q^2},$$

then it is a convergent of $\alpha$.

(A8) If two real numbers $\alpha_0$ and $\alpha_1$ coincide in the first n partial denominators of their regular continued fractions, then so do all real numbers inbetween $\alpha_0$ and $\alpha_1$.

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